Generalized functions - HW 3

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Question 1

We prove the claim: Assume (f_n) converges weakly to f. We will show that for every $x \in \mathbb{R}$ $f_n(x) \to f(x)$. Indeed, let δ_x be the translated δ-function. Since $C_c^\infty(\mathbb{R})$ is dense inside $C_c^{-\infty}(\mathbb{R}) \approx \overline{C_c^\infty(\mathbb{R})}^w$ there exists a sequence g_n^x which converges to δ_x in a distribution sense. That is for every $h \in C_c^\infty(\mathbb{R})$ we have that $< g_n^x,\ h > \to < \delta_x,\ h >= h(x)$. Now, consider the sequence $< f_n,\ g_n^x >$ since everything is linear and continuous and nice we can calculate it in two ways:

$$\lim f_n(x) =< \lim_n f_n,\ \delta_x >=< \lim_n f_n,\ \lim g_n^x >=< f,\ \lim g_n^x >=< f,\ \delta_x >= f(x)$$

The claim follows.

Question 2

Let $U = \bigcup_{i \in I} U_i,\ \xi_i \in C_c^\infty(U_i)$, and let $\{\varphi_i\}_{i \in I}$ be a partition of unity subordinate to $\{U_i\}$. Meaning, $\text{supp}(\varphi_i) \subset U_i$, for each $x \in \mathbb{R}$ only finitely many φ_i are not zero and $\sum_i \varphi = 1$. W.L.O.G we may assume each U_i is contained inside some compact set, otherwise we just refine the cover through compact exhaustion of the sets.

Since each φ_i is compactly supported, we may define the distribution $\varphi_i \xi_i$. and then define $\xi = \sum \varphi_i \xi_i$. Note that $\xi \in C_c^{-\infty}(U)$, and that for each $f \in C_c^\infty(U_i)$ we have that $< \xi \upharpoonright_{U_i},\ f >=< \xi,\ \bar{f} >$ where $\bar{f}$ is the extension by zeros of f into U. By linearity, $< \xi,\ \bar{f} >= \sum_j < \varphi_j \xi_j,\ \bar{f} >$ Since $\bar{f}$ is supported on U_i and all ξ_j agree with ξ_i on $U_i \cap U_j$ we may rewrite the sum as $< (\sum_j \varphi_j)\xi_i, \bar{f} >=< 1 \cdot \xi_i, f >$ and $\xi \upharpoonright_{U_i} = \xi_i$ as required.

Question 3

a) Before starting, we will prove the existence of linear functionals on $C_c^\infty(\mathbb{R})$ which are not continuous. Consider the linear functional $\varphi(f) = \sum_i \frac{1}{i!} f^{(i)}(0)$, it is clearly linear and according to Taylor's theorem it is also well defined. However, it does not depend on a finite number of derivatives, which is a converse to what we have seen for continuous functionals. Thus φ is a non-continuous functional.

To see that $C_c^{-\infty}(\mathbb{R})$ is not complete we consider it a subspace of $C_c^\infty(\mathbb{R})'$, the space of all linear functionals (not necessarily continuous). As we will show in the next item this space is complete, and by the previous claim it does not equal $C_c^{-\infty}(\mathbb{R})$. Thus, it will be enough to show that $C_c^{-\infty}(\mathbb{R})$ is dense inside $C_c^{\infty}(\mathbb{R})'$.

Indeed, let $\emptyset \neq U_{A,\{V_f\}} = \{\varphi \in C_c^{\infty}(\mathbb{R})' : |\varphi(f)| \in V_f, \forall f \in A\}$ be a basic open set where A is a finite set and $\{V_f\}_{f\in A}$ are open in $\mathbb{R}$. Let $\varphi \in U_{A,V}$ We define the functional $\psi(f) = \phi(f)$ if $f \in \text{span}(A)$ and 0 otherwise. Clearly, $f \in U_{A,V}$ and since f is bounded it is also linear (f is also well defined since A is finite). Thus, $C_c^{-\infty}(\mathbb{R})$ is dense inside $C_c^{\infty}(\mathbb{R})'$ which implies that it is not a closed subspace. Since $C_c^{\infty}(\mathbb{R})'$ is complete this implies that $C_c^{-\infty}(\mathbb{R})$ is not.

b)We will use a similar tactic to show that $C_c^{\infty}(\mathbb{R})'$ is complete. Only, this time we consider it as subspace of $\mathbb{R}^{C_c^{\infty}(\mathbb{R})}$, the space of all functions (not necessarily linear of continuous) with the weak topology. That, is the basic open sets are precisely $U_{A,\{V_f\}} = \{\varphi \in \mathbb{R}^{C_c^{\infty}(\mathbb{R})} : |\varphi(f)| \in V_f, \forall f \in A\}$. But these are precisely the basic open sets in the product topology on $\mathbb{R}^{C_c^{\infty}(\mathbb{R})}$. Thus, the weak topology and the product topology are the same. Since R is complete any product of it is also complete and $\mathbb{R}^{C_c^{\infty}(\mathbb{R})}$ is complete. It will suffice to show that $C_c^{\infty}(\mathbb{R})'$ is closed inside $\mathbb{R}^{C_c^{\infty}(\mathbb{R})}$. Equivalently, we will show that its complement is open.

Let $\varphi \in \mathbb{R}^{C_c^{\infty}(\mathbb{R})} \setminus C_c^{\infty}(\mathbb{R})'$. There are two options:

- There exists a function f and a scalar $\lambda \neq 1$ such that $\varphi(\lambda f) - \lambda\varphi(f) = \varepsilon > 0$. In that case, the basic open set which is $(\varphi(\lambda f) - \frac{\varepsilon}{2}, \varphi(\lambda f) + \frac{\varepsilon}{2})$ on λf, $(\varphi(f) - \frac{\varepsilon}{2}, \varphi(f) + \frac{\varepsilon}{2})$ on f and $\mathbb{R}$ at every other point must contain φ but cannot contain any linear function.

- The second option is that there are two function f and g such that $\varphi(f+g) \neq \varphi(f)+\varphi(g)$. In which case we repeat the same trick as above only this time we consider 3 functions: f, g and $f+g$.

Overall, for every φ as above there is an open neighborhood which is disjoint from $C_c^{\infty}(\mathbb{R})'$ which concludes the proof.

Question 4

a) We denote $U = \mathbb{R}^n \setminus \mathbb{R}^k$. Since $C_c^{\infty}(\mathbb{R}^n)$ is a limit of Frechet spaces we may describe $\overline{C_c^{\infty}(U)}$ as the set of all limits of sequences inside $C_c^{\infty}(U)$. It is not evident that $\overline{C_c^{\infty}(U)} \subset \bigcap_{m=1}^{\infty} V_m$. Since for each element of a sequence in $C_c^{\infty}(U)$ all derivatives must vanish in a neighborhood of $\mathbb{R}^k$ in the limit all derivatives must vanish in $\mathbb{R}^k$.

To see that $\bigcap_{m=1}^{\infty} V_m \subset \overline{C_c^{\infty}(U)}$ let $f \in \bigcap_{m=1}^{\infty} V_m$, we will find a sequence inside $C_c^{\infty}(U)$ which converges to f. To find such a sequence, consider the bump functions $\eta_i : \mathbb{R}^n \to [0,1]$ such that $\eta_i(\mathbb{R}^k) = 1$ and η_i vanishes outside of $B(\mathbb{R}^k, \frac{1}{i}) = \{x \in \mathbb{R}^n : \|x-y\| > \frac{1}{i}, \forall y \in \mathbb{R}^k\}$. We now define $\bar{\eta_i} = 1 - \eta_i$. Now, the sequence $(f \cdot \bar{\eta_i})$ is clearly inside $C_c^{\infty}(U)$, since each such function vanishes on a neighborhood of $\mathbb{R}^k$. On the other hand, if we define the compact set $K = \text{supp}(f)$ then all derivatives of $f \cdot \bar{\eta_i}$ converge uniformly to f in K. Thus, $f \cdot \bar{\eta_i} \to f$.

b) We consider the functional $\frac{\partial \delta}{\partial_{x_{k+1}}}$. If there is any justice in this world, its support is contained in $\mathbb{R}^k$. On the other hand, observe the function $f(x) = x_{k+1} \cdot \varphi$ where φ is some bump function supported on a neighborhood of the origin. Clearly $f \in V_m$ for every m since it does

not have any derivatives in the first k coordinates.
On the other hand, $< \frac{\partial \delta}{\partial_{x_{k+1}}}, f >= - < \delta, \varphi + x_{k+1} \cdot \frac{\partial \varphi}{\partial_{x_{k+1}}} >= -1 \neq 0$. So, $\frac{\partial \delta}{\partial_{x_{k+1}}} \notin \bigcup F_m$ as desired.

c) Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism such that $\varphi(\mathbb{R}^k) = \mathbb{R}^k$ and let $\xi \in F_m$ we would like to show that $\varphi(\xi) \in F_m$ as well. Where $< \varphi(\xi), f >=< \xi, f \circ \varphi >$. That is, if will suffice to show that if $f \in V_m$ then so is $f \circ \varphi$.

Clearly, $f \circ \varphi \in V_0$. Note that by the chain rule we may write the total differential as $D(f \circ \varphi) = Df \cdot D\varphi$. The condition on f assures that on $\mathbb{R}^k$, Df is a block matrix where all but the last $(n-k)X(n-k)$ coordinates are zero. Also, Since $\varphi(\mathbb{R}^k) = \mathbb{R}^k$ we know that on $\mathbb{R}^k$, $D\varphi$ is a block matrix with one block on the first kXk coordinates and another on the last $(n-k)X(n-k)$ coordinates. Thus $Df \cdot D\varphi$ is again zero on all but the last $(n-k)X(n-k)$ coordinates, and $f \circ \varphi \in V_1$. We may now proceed by induction, while viewing $D(f \circ \varphi)$ as a function on a higher dimensional space.

Question 5

Since V is finite dimensional, we may assume W.L.O.G. that $V \approx \mathbb{R}^k$. Let $\{e_i\}_{i=1}^k$ be the standard basis of V. Let $\varphi \in C_c^\infty(\mathbb{R}^n, V^*)^*$, we may think about φ as $\sum_{i=1}^k \varphi_i \cdot e^i$, where $\{e^i\}$ is the dual basis.

We now have a bijection from $C_c^\infty(\mathbb{R}^n, V^*)^*$ to $C_c^\infty(\mathbb{R}^n)^* \otimes V$ by $\varphi \leftrightarrow \sum_{i=1}^k \varphi_i \otimes e_i$.